Tutorial 5

Solving matrix games

Two useful principles: 1. Deleting the dominated rows and columns to obtain a new matrix with lower dimensions. Recall that a row is dominated if it is dominated (or say bounded) from above by another row, a column is dominated if it is dominated from below by another column.

2. The principle of indifference. Assume $\boldsymbol{p} = (p_1, \cdots, p_m)$ and $\boldsymbol{q} = (q_1, \cdots, q_n)$ are optimal strategies for Player I and Player II respectively. Then

(i) for any $k \in \{1, \cdots, m\}$ with $p_k > 0$, we have $\sum_{j=1}^{n} a_{k,j} q_j = v(A)$.

(ii) for any $l \in \{1, \cdots, n\}$ with $q_l > 0$, we have $\sum_{i=1}^{m} a_{i,l} p_i = v(A)$.

Recall the Minimax Theorem. This existence theorem also gives a characterization of the value of a game matrix and optimal strategies for the two players. More precisely, given an $m \times n$ matrix A, we call a number v the value of A, a probability vector $\boldsymbol{p} \in \mathcal{P}^m$ a maximin strategy for the row player, and a probability vector $\boldsymbol{q} \in \mathcal{P}^n$ a minimax strategy for the column player if

(i) $\boldsymbol{p} A \boldsymbol{y}^T \geq v$ for any $\boldsymbol{y} \in \mathcal{P}^n$.

(ii) $\boldsymbol{x} A \boldsymbol{q}^T \leq v$ for any $\boldsymbol{x} \in \mathcal{P}^m$.

(iii) $\boldsymbol{p} A \boldsymbol{q}^T = v$.

We note condition (i) is equivalent to

(i)$'$ every element of the row vector $\boldsymbol{p}A$ is at least v,

and the condition (ii) is equivalent to

(ii)$'$ every element of the column vector $A\boldsymbol{q}^T$ is at most v.

Exercise 1. *Let*

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

(i) Suppose $\lambda_1 \leq 0$ *and* $\lambda_n > 0$. *Find the value of* A.

(ii) Suppose $\lambda_1 > 0$. *Solve the two-person zero-sum game with game matrix* A.

Solution. (i) Let $k \geq 1$ be the smallest integer such that $\lambda_k \leq 0$ and $\lambda_{k+1} > 0$. Set $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{P}^n$ by

$$\boldsymbol{p} = (0, \cdots, 0, p_{k+1}, \cdots, p_n), \ \boldsymbol{q} = (q_1, \cdots, q_k, 0, \cdots, 0).$$

Then by the choice of k, we have

(a) $\boldsymbol{p}A = (0, \cdots, 0, p_{k+1}\lambda_{k+1}, \cdots, p_n\lambda_n)$ has all elements ≥ 0.

(b) $A\boldsymbol{q}^T = (q_1\lambda_1, \cdots, q_k\lambda_k, 0, \cdots, 0)^T$ has all elements ≤ 0.

(c) $\boldsymbol{p}A\boldsymbol{q}^T = 0$.

Hence by the Minimax Theorem, the value of A equals 0.

(ii) Let v denote the value of A. Assume $\boldsymbol{p} = (p_1, \cdots, p_n)$ is an optimal strategy for the row player. Assume $p_k > 0$ for $1 \leq k \leq n$. Then by the principle of indifference, we have

$$\begin{pmatrix} p_1 & \cdots & p_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \begin{pmatrix} v & \cdots & v \end{pmatrix},$$

which implies (since $\lambda_k > 0$ for all k)

$$v = \frac{1}{\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n}},$$

and

$$p_k = \frac{\frac{1}{\lambda_k}}{\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n}}, \text{ for } k = 1, \cdots, n.$$

Assume $\boldsymbol{q} = (q_1, \cdots, q_n)$ ($q_k > 0$ for all k) is an optimal strategy for the column player, it is easy to see $\boldsymbol{p} = \boldsymbol{q}$. Clearly, for the above v, $\boldsymbol{p}$ and $\boldsymbol{q}$, the conclusion of the Maximin Theorem holds. Hence $v, \boldsymbol{p}, \boldsymbol{q}$ are desired.

Exercise 2. *Player I and Player II choose integers i and j respectively from the set $\{1, \cdots 7\}$. Player I winns 1 dollar if $|i - j| = 1$, otherwise there is no payoff. Find the game matrix and solve the game.*

Solution. The game matrix is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

By deleting dominated rows and columns, we obtain the reduced matrix

$$A' = \begin{array}{c} \\ 2 \\ 3 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

By the principle of indifference, it is easy to see the value of A is $\frac{1}{4}$, and $(0, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}, 0)$ is the only optimal strategy for the row player, $(\frac{1}{4}, \frac{1}{4}, 0, 0, 0, \frac{1}{4}, \frac{1}{4})$ is the only optimal strategy for the column player.